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Statistical mechanics of the mixed majority–minority game with random external information

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Abstract

We study the asymptotic macroscopic properties of the mixed majority–minority game, modelling a population in which two types of heterogeneous adaptive agents, namely 'fundamentalists' driven by differentiation and 'trend-followers' driven by imitation, interact. The presence of a fraction f of trend-followers is shown to induce (a) a significant loss of informational efficiency with respect to a pure minority game (in particular, an efficient, unpredictable phase exists only for $f < 1/2$), and (b) a catastrophic increase of global fluctuations for $f > 1/2$. We solve the model by means of an approximate static (replica) theory and by a direct dynamical (generating functional) technique. The two approaches coincide and match numerical results convincingly.

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1. Introduction

In recent years, a substantial amount of research has been focused on model systems of heterogeneous adaptive agents interacting competitively, as e.g. in games, markets or ecosystems, in the attempt to understand the mechanisms by which real systems create exploitable information, and to clarify the origin of their complex collective behaviour [1]. The minority game, with its several variants, is perhaps the most studied of such models [2]. In its simplest version, it describes a population of inductive players with fully heterogeneous beliefs who, at each round of the game, make their strategic decisions on the basis of some public information pattern (the 'state of the world') aiming to be in the minority group. The minority-wins mechanism, which originally served the purpose of modelling competition for a scarce resource, translates into a strong assumption on the behavioural traits and expectations of players. Indeed, it turns out that in order to maximize their expected utilities under the minority rule, agents have to enhance their initial heterogeneity and differentiate themselves as much as possible from each other. This is rather intuitive: if agents were to learn to make decisions similarly to each other, being in a minority would become a rather unlikely event. On the other hand, one might also consider another tendency that is often encountered in real agents, namely that towards imitation, say of an agent who believes that his/her payoff is maximized when he/she acts according to the majority. In this paper, we consider a mixed majority–minority game, to study the effects of competition in a population formed by two types of players, i.e. those whose short-term behaviour is driven by imitation (who play a majority game), and those who are instead anti-imitative (and play a minority game).

From the viewpoint of economic modelling, our system represents a simple abstraction for a market where two classes of economic agents, namely 'fundamentalists' and 'trend-followers', interact. The former—see [3, 4] for details—create their expectations under the assumption that the market price is close to its 'fundamental' value, i.e. to a stationary equilibrium, and correspond to minority game players. The latter, instead, extrapolate a trend from recent price increments and assume that the next increment will occur in the direction of the trend (see also [5, 6]); they correspond to majority game players. In real markets, fundamentalists act as a kind of elastic force that pulls the price towards its fundamental value, while trend-followers destabilize the market by driving the price away from it. They are in particular widely believed to be the main actors in the infamous buy rushes known as 'bubbles'. Modelling the interplay between trend-followers and fundamentalists is a basic issue in the theory of markets, and several models have been proposed (see e.g. [5–8] and references therein). In most cases, however, an insight can be gained only from numerical simulations due to the complexity of the microscopic definitions. The minority game provides a basic framework for a tractable class of models (for another minority-game-based market model with two different types of agents, 'speculators' and 'producers', see [9]). The mixed model we consider here has indeed the advantage of being simple enough to be analytically solvable via the methods of statistical mechanics, notwithstanding its phenomenological richness. In particular, the effects due to the presence of trend-followers, namely a loss of informational efficiency and an increase of fluctuations, are fully discernible.

From a strictly theoretical viewpoint, the majority game is an intriguing model in itself, that shares some features with the Hopfield model of neural networks [10]. Surprisingly, it has not received much attention so far [11]. We will see that, at odds with what happens in the minority game, all agents taking part in a majority game actually manage to find a best strategy among those at their disposal and stick to it. One would hence be tempted to question the necessity of a multi-strategy, minority-game-like setup for a majority game. It will however become clear, particularly from the dynamical calculation, that such a 'freezing' is highly non-trivial and requires an in-depth analysis. Other interesting features of the majority game, such as the existence of a phase with finite excess demand in the presence of a particular state of the world (the analogue of a 'retrieval' phase in neural networks) are not considered here but certainly deserve attention. Some are the subject of [12].

This work is organized as follows. The basic definitions of the model are given in section 2, together with an outline of the results. The static approximation to the analysis of the asymptotic macroscopic properties is expounded in section 3. It is based on the formal analogy with zero-temperature spin glasses first derived in [13] for the pure minority game, whose stationary states were shown to be (approximately) given by the minima of a random Hamiltonian. In our case, the resulting optimization problem is slightly more subtle and its solution requires a negative dimensional replica theory of the kind already used for 'minimax games' [14], close in spirit to the method of partial annealing [15]. Section 4 is devoted to the dynamical solution of the 'batch' version of the model, which is carried out employing the generating functional technique [16] along the lines of [17, 18]. Some details about this calculation are given in the appendix. Finally, in section 5, we formulate our conclusions.

2. Definitions and outline of the results

The setup we consider is as follows. There are N players and P possible information patterns. For each player $i \in \{1, \ldots, N\}$ two strategies $\boldsymbol{a}_{ig} : \{1, \ldots, P\} \ni \mu \to a_{ig}^{\mu} \in \{-1, +1\}$ are given ($g = 1, 2$) that map an information pattern μ into a binary trading action a_{ig}^{μ} ('buy/sell'). (The generalization to S strategies per agent is possible but it is analytically less convenient.) We assume as usual that P scales with N so that $P/N = \alpha$ remains finite in the relevant limit $N \to \infty$ and that each a_{ig}^{μ} is selected randomly with uniform probability in $\{-1, 1\}$ at the beginning of the game for all i, μ and g and fixed. Strategies are evaluated according to their 'performance' $p_{ig}(n)$. At each round n, players receive an information pattern $\mu(n)$ chosen at random with uniform probability in $\{1, \ldots, P\}$ [19, 20]. Subsequently, each player picks his so-far best-performing strategy, $\widetilde{g}_i(n) = \arg\max_g p_{ig}(n)$, and formulates the bid it prescribes, i.e. $a_{i\widetilde{g}_i(n)}^{\mu(n)}$. The aggregate action of all players at round n (in economic terms, the 'excess demand') is just

$$A(n) = \frac{1}{\sqrt{N}} \sum_{i=1,N} a_{i\widetilde{g}_i(n)}^{\mu(n)}. \tag{1}$$

Once $A(n)$ is known, majority (resp. minority) game players reward their strategies for which $a_{ig}^{\mu(n)} A(n) > 0$ (resp. $a_{ig}^{\mu(n)} A(n) < 0$). Hence the performance updating or learning process takes place according to[1]

$$p_{ig}(n+1) - p_{ig}(n) = \epsilon_i a_{ig}^{\mu(n)} A(n) \qquad (g = 1, 2) \tag{2}$$

where $\epsilon_i = -1$ for minority game players and $\epsilon_i = +1$ for majority game players, and the game moves into the next round. The ϵ_i can be seen as an additional family of quenched r.v. (besides the a_{ig}^{μ}) with probability density $P(\epsilon_i) = f\delta_{\epsilon_i,+1} + (1-f)\delta_{\epsilon_i,-1}$.

It is convenient to introduce the 'preferences' $y_i(n) = (p_{i1}(n) - p_{i2}(n))/2$ and the quantities $\xi_i^{\mu} = \left(a_{i1}^{\mu} - a_{i2}^{\mu}\right)/2$, $\omega_i^{\mu} = \left(a_{i1}^{\mu} + a_{i2}^{\mu}\right)/2$ and $\Omega^{\mu} = N^{-1/2}\sum_{i=1,N}\omega_i^{\mu}$, which measure the degree to which each agent's strategies are similar or dissimilar. Using these, (2) can be recast as an equation for $y_i(n)$:

$$y_i(n+1) - y_i(n) = \epsilon_i \xi_i^{\mu(n)} \left[\Omega^{\mu(n)} + \frac{1}{\sqrt{N}} \sum_{j=1,N} \xi_j^{\mu(n)} s_j(n) \right] \tag{3}$$

where $s_i(n) = \mathrm{sign}[y_i(n)]$. When $y_i(n) > 0$ (resp. $y_i(n) < 0$) agent i selects strategy $g = 1$ (resp. $g = 2$) and $s_i(n) = +1$ (resp. $s_i(n) = -1$). As in the pure minority game, this stochastic (Markovian) dynamics is a zero-temperature process that violates detailed balance, so that strictly speaking it has no Lyapunov function.

One is interested in characterizing the macroscopic ($N \to \infty$) properties of the stationary state (if any exists) of (3). Two quantities have been introduced with this aim. As a measure of global efficiency we take the 'volatility'

$$\sigma^2 = \langle A^2 \rangle = \lim_{T\to\infty} \frac{1}{T - T_{\mathrm{eq}}} \sum_{n=T_{\mathrm{eq}},T} A(n)^2 \tag{4}$$

that is, the magnitude of market fluctuations (it can be shown that $\langle A \rangle = 0$ when $N \to \infty$). Intuitively, the higher the efficiency the smaller the σ^2. As a reference value, it is reasonable to

[1] We assume that players ignore their market impact, i.e. that they behave as price takers [21].

take $\sigma^2 = 1$, which corresponds to 'random players' who at each round randomize uniformly between the two possible actions. When $\sigma^2 < 1$ one can say that agents are, to some degree, cooperating. From the viewpoint of information creation, the relevant quantity is instead the 'predictability' or 'available information'

$$H = \frac{1}{P}\sum_{\mu=1,P}\langle A|\mu\rangle^2 \qquad \text{with} \quad \langle A|\nu\rangle = \lim_{T\to\infty}\frac{1}{T-T_{\text{eq}}}\sum_{n=T_{\text{eq}},T} A(n)\delta_{\mu(n),\nu} \tag{5}$$

whose meaning is discussed at length in the literature (see e.g. [21, 22]). The idea is that when $H > 0$ there exists at least one state of the world, say μ, such that $\langle A|\mu\rangle \neq 0$, i.e. for which there is an action that is more likely to be the winning action. An external agent entering the game could hence exploit this information to have a gain. The fact that $H > 0$ signals an inefficiency of the market. Regimes with $H > 0$ are dubbed 'asymmetric', at odds with 'symmetric' ones with $H = 0$ where the game's outcome is not predictable.

In the limit $N \to \infty$, σ^2 and H depend on α (as in the pure minority game) and f. Computer simulations of (3) suggest the following scenario (see figure 1). For $f < 1/2$, a minority-game type of behaviour is recovered, with an asymmetric phase ($H > 0$) at high α separated by a symmetric one ($H = 0$) at low α. The transition point α_c decreases as f increases, hence the symmetric phase shrinks as more and more trend-followers join the game, indicating that they provide an additional exploitable 'signal'. Market fluctuations tend to the random limit $\sigma^2 = 1$ for large α and decrease with α until the critical point is reached. In the sub-critical phase, the stationary state depends strongly on the initial conditions of (3), and both high-volatility and low-volatility states can be reached starting from slightly different configurations[2]. For $f > 1/2$, instead, trend-followers dominate the game and σ^2 decreases steadily with α and f. The market is asymmetric ($H > 0$) for all α and the difference between σ^2 and H diminishes as f increases. For $f = 1$, one has $\sigma^2 = H$. We found that σ^2 is practically independent of the initial conditions. The case $f = 1/2$ possesses some special features and will be treated separately [23]. Let it suffice to say that simulation results give $\sigma^2 \simeq 1$ with H showing a slow decrease with α. Unfortunately, reliable numerical experiments at $\alpha < 0.01$ are quite costly. So the precise analysis of this case requires additional studies. The theory we present here provides qualitative agreement with these experiments, but it is likely that in the $f = 1/2$ case a more refined analysis is possible.

In order to get some theoretical insight, one can follow the line of reasoning adopted for the pure minority game, for which it was shown by constructing the continuous-time limit of (3) that the average asymptotic value of s_i, denoted by m_i, can be obtained by minimizing the random function

$$\mathcal{H}(\boldsymbol{m}) = \frac{N}{P}\sum_{\mu=1,P}\left[\Omega^\mu + \frac{1}{\sqrt{N}}\sum_{i=1,N}\xi_i^\mu m_i\right]^2 \tag{6}$$

where $\boldsymbol{m} = \{m_i\}$. (Note that the m_i are 'soft' spins: $-1 \leqslant m_i \leqslant 1$. Note also that an expansion of the square in (6) ultimately clarifies that terms like $\sum_\mu \xi_i^\mu \xi_j^\mu$ play the role of random couplings while $\sum_\mu \xi_i^\mu \Omega^\mu$ acts as a random field.) We will not discuss here the limitations of this approximation and refer the reader to the original literature [13, 24–29] for a critical discussion. In the limit $N \to \infty$, this problem could be tackled using spin-glass techniques, because

$$\lim_{N\to\infty}\min_{\boldsymbol{m}}\frac{\mathcal{H}(\boldsymbol{m})}{N} = -\lim_{\beta\to\infty}\lim_{N\to\infty}\frac{1}{\beta N}\overline{\log Z(\beta)} \tag{7}$$

[2] Note, however, that if the initial conditions $y_i(0)$ contain a sufficiently large bias towards one of the strategies, all players will always use the same strategy, which will result in the 'random trading' state with $\sigma^2 = 1$.

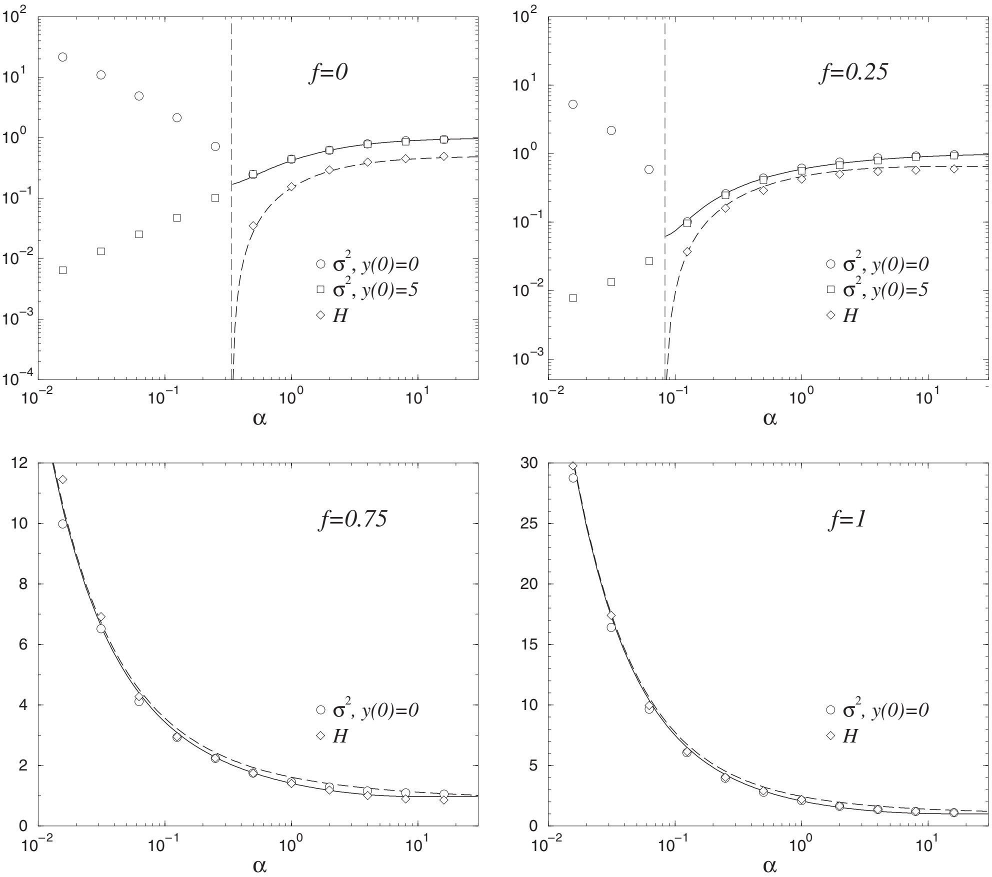

Figure 1. Behaviour of σ^2 and H versus α for $f = 0, 0.25, 0.75, 1$. Markers represent results from numerical simulations with homogeneous initial conditions, averaged over 200 disorder samples. The dashed vertical lines give the location of α_c (from theory). Continuous (resp. dashed) lines represent analytical approximations for σ^2 (resp. H), valid only for $\alpha > \alpha_c$. Results for H are compared with the static approximation of section 3, while those for σ^2 are compared with the dynamical results of section 4. The logarithmic scale on the y-axis in the upper panels has been used to stress the dependence of σ^2 on the initial conditions for $\alpha < \alpha_c$.

(here, $Z(\beta) = \int \mathrm{e}^{-\beta\mathcal{H}}\, \mathrm{d}\boldsymbol{m}$ and the over-line denotes an average over disorder, i.e. over different realizations of the agents' strategies). The evaluation of $\overline{\log Z}$ requires the replica trick [30]. For $\alpha > \alpha_c$, $\mathcal{H}$ has a unique minimum, hence the stationary state can be fully described by the replica-symmetric solution of (7).

This argument can be reformulated for the pure majority game. The corresponding optimization problem turns out to be

$$\max_{\boldsymbol{m}} \mathcal{H}(\boldsymbol{m}) \qquad \text{or equivalently} \qquad \min_{\boldsymbol{m}} -\mathcal{H}(\boldsymbol{m}). \tag{8}$$

A few comments are in order. First, it is easy to see that $H = \mathcal{H}/N$, which implies that minority game players roughly tend to minimize the available information, while majority ones tend to maximize it. Second, at odds with $\mathcal{H}$, $-\mathcal{H}$ possesses many minima, hence the stationary state of the majority game will always depend on the initial conditions of the dynamics (even though the macroscopic observables σ^2 and H might take on the same or very similar values in all minima). Based on well-known properties of the Hopfield model [10], one expects the true minima of $-\mathcal{H}$ to be described by solutions of (8) that break replica symmetry. Moreover, as happens in attractor neural networks with extensively many patterns, a 'retrieval' phase is to be expected for small enough α where, due to correlations between the initial conditions and one specific pattern, say $\mu = 1$, the overlap $o^\mu(\boldsymbol{m}) = N^{-1/2}\sum_{i=1,N}\xi_i^\mu m_i$ is $\mathrm{O}(N^{-1/2})$, and vanishing as $N \to \infty$, for all μ except $\mu = 1$, for which it is finite. The fact that agents can 'condense' around a given pattern implies that every time that pattern is presented to them a buy (or sell) rush takes place. Solving (8) is hence a non-trivial task in itself, and requires a detailed study [12].

Generalizing to our case, one finds that the stationary m_i for the mixed majority–minority model can be obtained by solving the following problem:

$$\max_{\boldsymbol{m}_2}\min_{\boldsymbol{m}_1}\mathcal{H}(\boldsymbol{m}_1, \boldsymbol{m}_2) \tag{9}$$

where $\boldsymbol{m}_1$ (resp. $\boldsymbol{m}_2$) denote collectively the m_i variables of minority (resp. majority) game players. Hence the mixed game where both minority and majority players are present at the same time requires a minimization of $\mathcal{H}$ in certain directions (the minority ones) and a maximization in others (the majority ones). Again, this problem can be tackled by a replica theory. The idea [14] is to introduce two 'inverse temperatures' β_1 and β_2 for minority and majority players respectively, such that

$$\max_{\boldsymbol{m}_2}\min_{\boldsymbol{m}_1}\mathcal{H}(\boldsymbol{m}_1, \boldsymbol{m}_2) = \lim_{\beta_1,\beta_2\to\infty}\frac{1}{\beta_2}\overline{\log\mathcal{Z}(\beta_1, \beta_2)} \tag{10}$$

with the following generalized partition function:

$$\mathcal{Z}(\beta_1, \beta_2) = \int \mathrm{d}\boldsymbol{m}_2\, \mathrm{e}^{\beta_2[-\frac{1}{\beta_1}\log\int\mathrm{d}\boldsymbol{m}_1\mathrm{e}^{-\beta_1\mathcal{H}}]} = \int \mathrm{d}\boldsymbol{m}_2\left[\int \mathrm{d}\boldsymbol{m}_1\,\mathrm{e}^{-\beta_1\mathcal{H}}\right]^{-\gamma} \tag{11}$$

where $\gamma = \beta_2/\beta_1 > 0$. In physical jargon, this describes a system where: first, the $\boldsymbol{m}_1$ variables are thermalized at a positive temperature $1/\beta_1$ with Hamiltonian $\mathcal{H}$ at fixed $\boldsymbol{m}_2$; then, the $\boldsymbol{m}_2$ variables are thermalized at a negative temperature $-1/\beta_2$ with an effective Hamiltonian $\mathcal{H}_{\mathrm{eff}}$ defined by $-\beta_1\mathcal{H}_{\mathrm{eff}}(\boldsymbol{m}_2) = \log\int\mathrm{d}\boldsymbol{m}_1\,\mathrm{e}^{-\beta_1\mathcal{H}}$. The disorder average can be carried out with the help of a 'nested' replica trick. First, one replicates the minority variables by treating the exponent $-\gamma$ as a positive integer R (at the end, the limit $R \to -\gamma < 0$ must be taken). Equation (11) thus becomes

$$\mathcal{Z} = \int \mathrm{d}\boldsymbol{m}_2\left[\int \mathrm{d}\boldsymbol{m}_1\,\mathrm{e}^{-\beta_1\mathcal{H}}\right]^R = \int \mathrm{d}\boldsymbol{m}_2\left[\int \mathrm{e}^{-\beta_1\sum_r\mathcal{H}(\{\boldsymbol{m}_1^r\},\boldsymbol{m}_2)}\prod_{r=1,R}\mathrm{d}\boldsymbol{m}_1^r\right]. \tag{12}$$

Then a second replication is needed (we remind the reader that replica theories use the fact that $\overline{\log Z} = \lim_{n\to0}(1/n)\log\overline{Z^n}$), this time on the $\boldsymbol{m}_2$ variables:

$$\mathcal{Z}^n = \int \mathrm{e}^{-\beta_1\sum_{a,r}\mathcal{H}(\{\boldsymbol{m}_1^{ar}\},\{\boldsymbol{m}_2^a\})}\prod_{a=1,n}\prod_{r=1,R}\mathrm{d}\boldsymbol{m}_1^{ar}\,\mathrm{d}\boldsymbol{m}_2^a. \tag{13}$$

At this point we have two replica indices with different roles: the a replicas have been introduced to deal with the disorder, and their number n will eventually go to zero, as usual; the r replicas have been introduced to deal with the negative temperature, and their number R must be set to a negative value. This kind of limit is not completely new in replica theories; it is what is usually done for example to express determinants via a bosonic integral representation (see for instance [31] for a discussion and [32] for an application). Majority variables bear just one index, while minority ones have two. We can interpret this fact by saying that $\boldsymbol{m}_2^a$ indicates a particular configuration of the majority variables, i.e. a given manifold in the whole $\boldsymbol{m}$ space; and $\boldsymbol{m}_1^{ar}$ indicates the minority coordinates in that particular manifold.

In section 3 we will solve (10) in the limit $N \to \infty$ using (13) as a starting point. Retrieval solutions for the majority part become increasingly important as f gets bigger. We will however neglect this aspect (which in the mixed case leads to a serious lengthening) completely. As retrieval requires that the system is prepared in specific initial conditions, one can say that we study the evolution from generic initial states. Results obtained in this way give a very good agreement with numerical simulations. It is important to say that macroscopic quantities such as the volatility might be different in a retrieval situation. And, of course, the latter is expected to play a very important role for phenomena that are local in time (like 'bubbles'). Besides this static approximation, we will also tackle the dynamics (3) straightforwardly, resorting to the generating-functional method to carry out the disorder-average [16]. Again, we will neglect the possibility of retrieval. Following [17], we will focus on the 'batch' version of the model. Dynamical results obtained in this way turn out to coincide nicely with their static counterpart and suggest that the transition occurring at α_c for $f < 1/2$ is related essentially to the onset of anomalous response, as in the pure minority game. We will calculate the critical line $\alpha_c(f)$, showing that $\alpha_c \downarrow 0$ as $f \uparrow 1/2$. For $f > 1/2$, the response is always finite and the macroscopic properties are dominated by the contribution of trend-followers.

3. Statics

To begin with, let us re-write the Hamiltonian (6) as

$$\mathcal{H}(\boldsymbol{m}_1, \boldsymbol{m}_2) = \frac{1}{P} \sum_{\mu=1,P} \left[\sum_{i=1,N} \omega_i^\mu + \sum_{j \in N_1} \xi_j^\mu m_{1j} + \sum_{k \in N_2} \xi_k^\mu m_{2k} \right]^2 \tag{14}$$

where N_1 (resp. N_2) denotes both the set and the cardinality of the set of minority (resp. majority) game players. The replicated Hamiltonian entering (13) is

$$\mathcal{H}\big(\{\boldsymbol{m}_1^{ar}\}, \{\boldsymbol{m}_2^{a}\}\big) = \frac{1}{P} \sum_{\mu=1,P} \left[\sum_{i=1,N} \omega_i^\mu + \sum_{j \in N_1} \xi_j^\mu m_{1j}^{ar} + \sum_{k \in N_2} \xi_k^\mu m_{2k}^{a} \right]^2. \tag{15}$$

We can linearize the exponential in (13) via a Hubbard–Stratonovich transformation introducing some auxiliary Gaussian variables z_{ar}^μ. Subsequently, the average over the disorder can be performed using the distribution $P\big(a_{ig}^\mu\big) = 1/2\big(\delta_{a_{ig}^\mu,1} + \delta_{a_{ig}^\mu,-1}\big)$ $(g = 1, 2)$ and the definitions of ω_i^μ and ξ_i^μ. One obtains

$$\begin{aligned}\overline{\mathcal{Z}^n} = \int \left[\prod_{a,r} \mathrm{d}\boldsymbol{m}_1^{ar}\, \mathrm{d}\boldsymbol{m}_2^{a}\right] \left[\prod_{\mu,a,r} \frac{\mathrm{d}z_{ar}^\mu}{\sqrt{2\pi}}\right] \exp\left[-\sum_\mu \sum_{ar} \frac{\big(z_{ar}^\mu\big)^2}{2}\right] \exp\Bigg[-\frac{\beta_1}{2\alpha} \sum_\mu \sum_{abrs} z_{ar}^\mu z_{bs}^\mu \\ \times \left(1 + (1-f)\frac{1}{N_1}\sum_{j\in N_1} m_{1j}^{ar} m_{1j}^{bs} + f\frac{1}{N_2}\sum_{k\in N_2} m_{2k}^{a} m_{2k}^{b}\right)\Bigg].\end{aligned} \tag{16}$$

It is now convenient to define the overlaps

$$Q_{ar,bs} = \frac{1}{N_1} \sum_{j \in N_1} m_{1j}^{ar} m_{1j}^{bs} \qquad \text{and} \qquad P_{ab} = \frac{1}{N_2} \sum_{k \in N_2} m_{2k}^{a} m_{2k}^{b} \tag{17}$$

inserting them in (16) via δ-distributions with Lagrange multipliers $\widehat{Q}_{ar,bs}$ and $\widehat{P}_{ab}$. Note that the overlap matrices Q and P are nR-dimensional and n-dimensional, respectively. In this way the site dependence can be dealt with, so that after a little algebra one gets (all numerical factors are 'hidden' in the $D(\cdot,\cdot)$ shorthand)

$$\overline{\mathcal{Z}^n} = \int \mathrm{e}^{NS(\mathsf{Q},\widehat{\mathsf{Q}},\mathsf{P},\widehat{\mathsf{P}})} D(\mathsf{Q},\widehat{\mathsf{Q}}) D(\mathsf{P},\widehat{\mathsf{P}}) \tag{18}$$

where the effective action S is given by $(a, b = 1, \ldots, nR; r, s = 1, \ldots, R)$

$$\begin{aligned} S(\mathsf{Q},\widehat{\mathsf{Q}},\mathsf{P},\widehat{\mathsf{P}}) = &-\frac{\alpha}{2}\log\det\mathsf{T} - \mathrm{i}[(1-f)\,\mathrm{Tr}(\widehat{\mathsf{Q}}\mathsf{Q}) + f\,\mathrm{Tr}(\widehat{\mathsf{P}}\mathsf{P})] \\ &+ (1-f)\log\int_{-1}^{+1}\left[\prod_{a,r}\mathrm{d}m_1^{a,r}\right]\exp\left[\mathrm{i}\sum_{abrs} m_1^{ar}\widehat{Q}_{ar,bs}m_1^{bs}\right] \\ &+ f\log\int_{-1}^{+1}\left[\prod_{a}\mathrm{d}m_2^{a}\right]\exp\left[\mathrm{i}\sum_{ab} m_2^{a}\widehat{P}_{ab}m_2^{b}\right] \end{aligned}$$

with

$$\mathsf{T} = \mathsf{I}_{nR} + \frac{\beta_1}{\alpha}[\mathsf{E}_{nR} + (1-f)\mathsf{Q} + f\mathsf{P}\otimes\mathsf{E}_R]. \tag{19}$$

I_K stands for the $K\times K$ identity matrix while E_K denotes the $K\times K$ matrix with all elements equal to 1. $\otimes$ is the Kronecker product. In (19) one can easily recognize some parts coming from the minority agents (those proportional to $(1-f)$) and others coming from the majority agents. These contributions are *not* factorized (in that event, the mixed problem would be trivial) but are interconnected via the determinant of T.

To proceed further, one has to formulate ansätze for the overlap matrices and then perform the integral (18) in the limit $N\to\infty$ by the steepest descent method. Let us first arrange Q in a convenient matrix form. We choose to order the indices in such a way that each row is characterized by a couple (a, r); along the row, the index a is first kept fixed while r varies from 1 to R. Q is thus naturally subdivided into blocks of size $R\times R$, the blocks along the diagonal corresponding to a given value of $a = b$. We recall that keeping a fixed corresponds to selecting, in the global configuration space, a well-defined manifold with $\boldsymbol{m}_2 = \boldsymbol{m}_2^a$ inside which $\mathcal{H}$ is minimized with respect to the $\boldsymbol{m}_1$ variables. $Q_{ar,as}$ can thus be interpreted as the overlap between two configurations of the same constrained minority problem. It is natural to assume for these diagonal sub-blocks the same matrix structure of a pure minority game, that is a symmetric form with a diagonal element Q and an off-diagonal one q_1. On the other hand, elements of the type $Q_{ar,bs}$ with $a\neq b$ correspond to overlaps between two minority configurations in different majority manifolds, and the simplest choice one can make is to take $Q_{ar,bs} = q_0$ for all of them. In this way Q assumes what is called a one-step RSB (replica symmetry broken) form [30]:

$$Q_{ar,bs} = (Q-q_1)\delta_{ab}\delta_{rs} + (q_1-q_0)\epsilon_{arbs} + q_0 \tag{20}$$

where the tensor ϵ_{arbs} is equal to 1 in the diagonal $R\times R$ blocks with $a = b$, and 0 elsewhere. Note that, contrary to standard replica calculations, here the block size R is not a variational parameter, but its value is fixed by the nature of the problem. For consistency, we adopt the same ansatz for the conjugated matrix $\widehat{\mathsf{Q}}$. The choice for the $n\times n$ matrices P and $\widehat{\mathsf{P}}$ is on the other hand more straightforward: we will consider the simple replica-symmetric ansatz

$$P_{ab} = (P-p_0)\delta_{ab} + p_0 \tag{21}$$

and take an analogous form for $\widehat{\mathsf{P}}$.

Putting (20) and (21) into (19), and using the conventional re-scalings $\widehat{\mathsf{Q}} = \left(-\mathrm{i}\beta_1^2\alpha/2\right)\Omega$ and $\widehat{\mathsf{P}} = \left(-\mathrm{i}\beta_1^2\alpha/2\right)\mathsf{G}$, the 'free energy' density $F = -S/(\beta_1 n)$ turns out to be given, in the limit $n \to 0$, by

$$
\begin{aligned}
F = {} & \frac{\alpha R}{2\beta_1} \log\left[1 + (1-f)\frac{\beta_1}{\alpha}(Q - q_1)\right] + \frac{\beta_1 R\alpha(1-f)}{2}\,[\Omega Q + (R-1)\omega_1 q_1 - R\omega_0 q_0] \\
& + \frac{\alpha}{2\beta_1}\log\left[1 + R\beta_1 \frac{(1-f)(q_1 - q_0) + f(P - p_0)}{\alpha + (1-f)\beta_1(Q - q_1)}\right] + \frac{\beta_1\alpha}{2} f(GP - g_0 p_0) \\
& + \frac{\alpha R}{2}\frac{1 + (1-f)q_0 + f p_0}{[\alpha + (1-f)\beta_1(Q-q_1)]\,[R\beta_1(1-f)(q_1 - q_0) + f(P - p_0)]} \\
& - \frac{1-f}{\beta_1}\int \mathrm{d}z\,\mathcal{P}(z)\log\int \mathrm{d}y\,\mathcal{P}(y)\left[\int_{-1}^{1}\mathrm{d}m_1\,\mathrm{e}^{-\beta_1 V_{zy}(m_1)}\right]^R \\
& - \frac{f}{\beta_1}\int \mathrm{d}z\,\mathcal{P}(z)\log\int_{-1}^{1}\mathrm{d}m_2\,\mathrm{e}^{-\beta_1 V_z(m_2)}
\end{aligned} \tag{22}
$$

where $\mathcal{P}(x) = \mathrm{e}^{-x^2/2}/\sqrt{2\pi}$ and

$$V_{zy}(m_1) = -z\sqrt{\alpha\omega_0}\,y m_1\sqrt{\alpha(\omega_1 - \omega_0)} - \frac{\alpha\beta_1}{2}(\Omega - \omega_1)m_1^2 \tag{23}$$

$$V_z(m_2) = -\sqrt{\alpha g_0}\,z m_2 - \frac{\alpha\beta_1}{2}(G - g_0)m_2^2. \tag{24}$$

The replica recipe now prescribes an extremization of (22) with respect to its ten variational parameters (namely Q, q_0, q_1, P, p_0 and their conjugate variables), because when $N \to \infty$

$$\lim_{N\to\infty}\max_{m_2}\min_{m_1}\frac{\mathcal{H}}{N} = \lim_{\beta_1,\beta_2\to\infty}\frac{F(\text{saddle point})}{R}. \tag{25}$$

This leaves us with a set of ten equations in ten variables. Defining

$$\chi_1 = \frac{\beta_1}{\alpha}(Q - q_1) - \frac{\beta_2}{\alpha}(q_1 - q_0) \tag{26}$$

$$\chi_2 = \frac{\beta_2}{\alpha}(P - p_0) \tag{27}$$

$$\chi = (1-f)\chi_1 - f\chi_2 \tag{28}$$

and using the shorthand

$$\langle\!\langle\cdots\rangle\!\rangle = \int \mathrm{d}z\,\mathcal{P}(z)\left[\frac{\int \mathrm{d}y\,\mathcal{P}(y)\big[\mathcal{Q}^{R-1}\int_{-1}^{1}\mathrm{d}m_1\cdots\mathrm{e}^{-\beta_1 V_{yz}(m_1)}\big]}{\int \mathrm{d}y\,\mathcal{P}(y)[\mathcal{Q}^R]}\right] \tag{29}$$

$\mathcal{Q} = \int_{-1}^{1}\mathrm{d}m_1\,\mathrm{e}^{-\beta_1 V_{yz}(m_1)}$ being a normalization integral, and

$$\langle\cdots\rangle_2 = \frac{\int_{-1}^{1}\mathrm{d}m_2\cdots\mathrm{e}^{-\beta_1 V_z(m_2)}}{\int_{-1}^{1}\mathrm{d}m_2\,\mathrm{e}^{-\beta_1 V_z(m_2)}} \tag{30}$$

we find the following system:

$$Q = \langle\!\langle m_1^2\rangle\!\rangle \tag{31}$$

$$\beta_1 R q_1 + \beta_1(Q - q_1) = \frac{\langle\!\langle y m_1\rangle\!\rangle}{\sqrt{\alpha(\omega_1 - \omega_0)}} \tag{32}$$

$$\alpha\chi_1 = \frac{\langle\!\langle z m_1\rangle\!\rangle}{\sqrt{\alpha\omega_0}} \tag{33}$$

$$\beta_1(\Omega - \omega_1) = -\frac{1}{\alpha + \beta_1(1-f)(Q-q_1)} \tag{34}$$

$$\omega_1 - \omega_0 = \frac{(1-f)(q_1-q_0) + f(P-p_0)}{\alpha(1+\chi)[\alpha + \beta_1(1-f)(Q-q_1)]} \tag{35}$$

$$\omega_0 = \frac{1 + (1-f)q_0 + fp_0}{\alpha^2(1+\chi)^2} \tag{36}$$

$$P = \langle m_2^2 \rangle_2 \tag{37}$$

$$g_0 = R^2\omega_0 \tag{38}$$

$$\alpha\chi_2 = -R\frac{\langle zm_2\rangle_2}{\sqrt{\alpha g_0}} \tag{39}$$

$$\beta_1(G - g_0) = -\frac{R}{\alpha(1+\chi)}. \tag{40}$$

Some observations about these equations are in order. First, if we set $f = 0$ we recover exactly the saddle-point equations for a pure minority game problem at inverse temperature β_1. With regard to the χ, it will soon become clear that χ_1 is the susceptibility of minority agents and, when $f = 0$, it reproduces the susceptibility of a pure minority game, while χ_2 is the susceptibility of majority agents. On the other hand, χ is *not* the global susceptibility. This is a consequence of the fact that in order to treat minority and majority players within the same formalism we had to introduce the effective negative inverse temperature $-\beta_2$.

Solving the above system at finite temperature(s) is a quite difficult task. Fortunately, in this case we are only interested in the limit of zero temperature(s), in which the solution of (31)–(40) turns out not to depend explicitly on R, provided G and g_0 are rescaled by R^2. Specifically, we look for solutions with $q_0 \to q_1 \to Q$ and $p_0 \to P$ such that χ_1, χ_2 and χ remain finite. These assumptions are justified for minority variables by the existence of just one global minimum of $\mathcal{H}$ (which also means that the minimum is unique in each manifold with given $\boldsymbol{m}_2$). On the other hand, they are more questionable for majority variables, since the maxima of $\mathcal{H}$ are numerous and disconnected (they occur in the corner of the configuration space $[-1, 1]^N$). However, they are the simplest possible in the absence of retrieval states. We will adopt them for this reason, but it should be kept in mind that they may not be the most appropriate ones in general.

After some algebra, the set of saddle-point equations can be greatly simplified, because, as in [21], when $\beta_1, \beta_2 \to \infty$ the averages (29) and (30) can be explicitly performed by steepest descent. The result for the relevant quantities is

$$P = 1 \tag{41}$$

$$Q = 1 - \sqrt{\frac{2}{\pi}}\frac{\mathrm{e}^{-\frac{\lambda^2}{2}}}{\lambda} - \left(1 - \frac{1}{\lambda^2}\right)\mathrm{erf}\frac{\lambda}{\sqrt{2}} \tag{42}$$

$$\frac{\alpha\chi}{1+\chi} = (1-f)\,\mathrm{erf}\frac{\lambda}{\sqrt{2}} - f\sqrt{\frac{2}{\pi}}\lambda \tag{43}$$

with $\lambda = \sqrt{\alpha/[1 + (1-f)Q + f]}$. The identity $P = 1$ implies that majority agents use only one of their strategies, i.e. that the stationary state of a pure majority game is in pure strategies. We define

$$c = (1-f)Q + f. \tag{44}$$

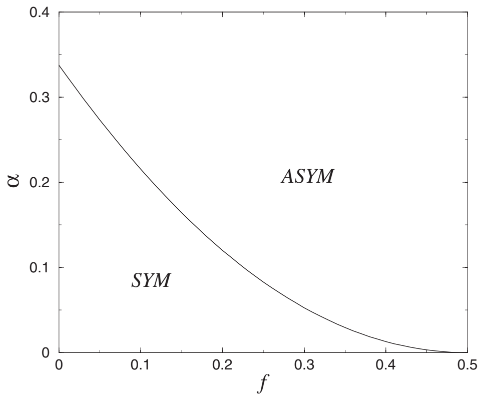

Figure 2. Critical line separating the asymmetric, inefficient phase with $H > 0$ from the symmetric one with $H = 0$ in the (f, α) plane. As $\alpha \downarrow \alpha_c(f)$, $\chi \to \infty$.

H can be expressed in terms of all saddle-point values since $\mathcal{H}/N = H$. Using (25) and taking the limit $R \to -1$ (this is equivalent to taking the limit $\beta_1 \to \beta_2$ followed by $\beta_2 \to \infty$) one finds

$$H = \frac{1+c}{2(1+\chi)^2}. \tag{45}$$

The existence of a transition at some critical value of α is determined by the divergence of χ (which means that H becomes 0). From (43) we find for α_c the following expression:

$$\alpha_c(f) = (1-f)\,\mathrm{erf}(x) - \frac{2fx}{\sqrt{\pi}} \tag{46}$$

where x is the solution of

$$2 - (1-f)\,\mathrm{erf}(x) - \frac{1-f}{x\sqrt{\pi}}\,\mathrm{e}^{-x^2} + \frac{f}{x\sqrt{\pi}} = 0. \tag{47}$$

Solving the above equations numerically for different f one obtains a very good agreement with the behaviour of H (see figure 1). The critical line α_c calculated from (46), (47) is instead displayed in figure 2. The symmetric phase shrinks upon increasing the fraction of majority agents and disappears for $f > 1/2$. It is interesting to note that a similar phenomenon was found in [9] upon increasing the number of 'producers'. Both majority agents here and producers in [9] use just one of their strategies, hence providing an exploitable signal to minority agents. This makes the market more and more informationally inefficient.

It should be mentioned that an approximate expression for σ^2 can also be obtained, $\sigma^2 \simeq H + (1-c)/2$, but it is not as accurate as that for H. A better estimate of σ^2 is obtained by solving the dynamics. As a last remark, let us note that for a pure majority game one gets, from (36)–(39) and from the fact that $\langle zm_2\rangle_2 = \sqrt{2/\pi}$,

$$\chi_2 = \frac{1}{1+\sqrt{\alpha\pi}} \qquad \text{and} \qquad H = (1-\chi_2)^{-2}. \tag{48}$$

The expression for χ_2 is identical (apart from a numerical factor) to that of the Hopfield model at zero temperature [33].

4. Dynamics

Let us turn our attention to the dynamics. For simplicity, we concentrate on the 'batch' case [17], which is obtained by averaging (3) over the μ and re-scaling time. This amounts to considering the case in which performance updates are made after many ($O(P)$) iterations rather than at the end of every round. One arrives at

$$y_i(t+1) - y_i(t) = \epsilon_i h_i + \epsilon_i \sum_{j=1,N} J_{ij} s_j(t) \tag{49}$$

where $h_i = (2/\sqrt{N}) \sum_{\mu=1,P} \xi_i^\mu \Omega^\mu$ and $J_{ij} = (2/N) \sum_{\mu=1,P} \xi_i^\mu \xi_j^\mu$. The stationary state of (49) is not identical to that of (3) (strictly speaking, not even for $N \to \infty$), but the corresponding macroscopic properties are qualitatively very similar. Moreover, the dynamical solution of (49) is significantly simpler than that of (3).

The dynamical approach consists in introducing a dynamical partition function of (49) as

$$Z[\psi] = \left\langle \mathrm{e}^{\mathrm{i} \sum_{it} y_i(t) \psi_i(t)} \right\rangle_{\text{paths}} \tag{50}$$

where the average is over the 'paths' $y_i(t)$ that satisfy (49) and the external sources $\psi_i(t)$ have been introduced for later convenience. Using the integral representation of the Dirac δ distribution to enforce the dynamical constraint (49), Z becomes

$$Z[\psi] = \int \mathrm{e}^{\mathrm{i} \sum_{it} \widehat{y}_i(t)[y_i(t+1) - y_i(t) - \epsilon_i h_i - \epsilon_i \sum_j J_{ij} s_j(t) - \theta_i(t)] + y_i(t)\psi_i(t)} p(\boldsymbol{y}(0)) D(\boldsymbol{y}, \widehat{\boldsymbol{y}}) \tag{51}$$

where the $\widehat{y}_i$ are the variables coming from the δ-function's integral representation, and $D(\boldsymbol{y}, \widehat{\boldsymbol{y}}) = \prod_{it} [\mathrm{d}y_i(t)\, \mathrm{d}\widehat{y}_i(t)/(2\pi)]$ and θ_i is a time-dependent external field. In principle, dynamical methods allow us to calculate disorder-averaged correlation and response functions exactly at all times by taking derivatives of the disorder-averaged Z, i.e.

$$\overline{Z[\psi]} = \int \mathrm{e}^{\mathrm{i} \sum_{it} \widehat{y}_i(t)[y_i(t+1) - y_i(t) - \theta_i(t)] + y_i(t)\psi_i(t) + NF(\widehat{\boldsymbol{y}})} p(\boldsymbol{y}(0)) D(\boldsymbol{y}, \widehat{\boldsymbol{y}}) \tag{52}$$

$$F(\widehat{\boldsymbol{y}}) = \frac{1}{N} \log \overline{\left[\mathrm{e}^{-\mathrm{i} \sum_{it} \widehat{y}_i(t) \epsilon_i [h_i + \sum_j J_{ij} s_j(t)]} \right]} \tag{53}$$

with respect to the fields ψ_i and θ_i [16, 34, 35], and have the advantage of not relying on the existence of a Lyapunov function. The evaluation of $\overline{Z}$ in the limit $N \to \infty$ leads to an effective (non-Markovian) process that provides an equivalent description of the original (Markovian) multi-agent process (49), and from which a closed set of equations for correlation and response functions can be derived. Here, we will be interested in the stationary solutions only. The calculation is in our case rather similar to that done for the pure batch minority game in [17], and is sketched in the appendix. The main difference is that here we obtain *two* effective processes, describing trend-followers and fundamentalists respectively. These are given by

$$y(t+1) - y(t) = \alpha\epsilon \sum_{t'} [(\mathsf{I} + \mathsf{G})^{-1}]_{tt'} s(t') + \theta(t) + \sqrt{\alpha} z(t) \tag{54}$$

where $\epsilon = 1$ (resp. -1) for the majority (resp. minority) part, and $z(t)$ is a zero-average Gaussian random variable with temporal correlations

$$\langle z(t) z(t') \rangle = [(\mathsf{I} + \mathsf{G})^{-1} (\mathsf{E} + \mathsf{C}) (\mathsf{I} + \mathsf{G}^T)^{-1}]_{tt'} \tag{55}$$

I stands for the identity matrix while E denotes the matrix with all elements equal to one. C has elements $C_{tt'} = \langle s(t) s(t') \rangle$. G, instead (see appendix for details), is the sum of two contributions:

$$\mathsf{G} = (1 - f)\mathsf{G}_1 - f\mathsf{G}_2. \tag{56}$$

G_1 (resp. G_2) has elements $\langle \partial s(t)/\partial\theta(t')\rangle_{-1}$ (resp. $\langle \partial s(t)/\partial\theta(t')\rangle_1$) where the subscript means average over the process (54) with $\epsilon = -1$ (resp. 1). When $N \to \infty$, $C_{tt'}$ can be identified with the disorder- and agent-averaged autocorrelation function of (49), while the two components of $G_{tt'}$ become identical to the disorder- and agent-averaged response functions of minority and majority agents, respectively.

Ergodic stationary states can be studied under the following assumptions:

- Time-translation invariance (TTI): $\begin{cases} \lim_{t\to\infty} C_{t+\tau,t} = C(\tau) \\ \lim_{t\to\infty} G_{t+\tau,t} = G(\tau) \end{cases}$;
- Finite integrated 'response' (FIR): $\lim_{t\to\infty}\sum_{t'\leqslant t} G_{tt'} = \chi < \infty$;
- Weak long-term memory (WLTM): $\lim_{t\to\infty} G(t,t') = 0\ \forall t'$ finite.

The breakdown of any of these signals the breakdown of ergodicity. To be clearer, we remark that the 'integrated response' χ defined in FIR has two components, i.e.

$$\chi = (1-f)\chi_1 - f\chi_2 \tag{57}$$

and can be negative. χ_1 and χ_2 are the actual susceptibilities of minority and majority agents, respectively. With FIR, we will require that *both* χ_1 and χ_2 are finite.

As in the minority game, for individual agents there are two possibilities: either $y_i(t)/t \to$ constant $\neq 0$ as $t \to \infty$, in which case they use only one of their strategies asymptotically (we call these agents 'frozen'); or $y_i(t)/t \to 0$ as $t \to \infty$, in which case they keep flipping between their strategies even in the long run (we call these agents 'fickle'). Macroscopic quantities can be obtained by separating the contributions of frozen and fickle agents.

Defining $\widetilde{y} = \lim_{t\to\infty} y(t)/t$, $s = \lim_{\tau\to\infty}(1/\tau)\sum_{t\leqslant\tau}\mathrm{sign}[y(t)/t]$ and $z = \lim_{\tau\to\infty}(1/\tau)\sum_{t\leqslant\tau} z(t)$, one has that

$$\widetilde{y} = \frac{\alpha\epsilon s}{1+\chi} + \sqrt{\alpha}z + \theta = \sqrt{\alpha}\epsilon\gamma s + \sqrt{\alpha}z + \theta. \tag{58}$$

Let us assume that $\gamma > 0$ (this assumption is verified *a posteriori*). For minority game players ($\epsilon = -1$), we have a frozen agent (with $s = \mathrm{sign}(\widetilde{y})$) if $|z| > \gamma$ and a fickle or non-frozen agent (with $s = z/\gamma$) if $|z| < \gamma$ [17]. In the majority part, all agents turn out to be frozen. In particular, for $z > \gamma$ agents freeze at $s = 1$, for $z < -\gamma$ they freeze at $s = -1$, while for $|z| < \gamma$ they can freeze at either value of s. It follows that the average autocorrelation $c = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t\leqslant\tau} C(t)$ is given by

$$\begin{aligned} c &= (1-f)\left[\langle\theta(|z|-\gamma)\rangle + \left\langle\theta(\gamma-|z|)\left(\frac{z}{\gamma}\right)^2\right\rangle\right] + f \\ &= (1-f)\left[1 - \mathrm{erf}\frac{\lambda}{\sqrt{2}} + \frac{1}{\lambda^2}\left(\mathrm{erf}\frac{\lambda}{\sqrt{2}} - \lambda\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\lambda^2/2}\right)\right] + f \end{aligned} \tag{59}$$

where we have separated the contributions of minority agents from majority agents, used the notation $\langle\,\rangle$ for the average over Gaussian r.v. z with variance

$$\langle z^2\rangle = \lim_{\tau,\tau'\to\infty}\frac{1}{\tau\tau'}\sum_{t\leqslant\tau,t'\leqslant\tau'}[(\mathsf{I}+\mathsf{G})^{-1}(\mathsf{E}+\mathsf{C})(\mathsf{I}+\mathsf{G}^T)^{-1}]_{tt'} = \frac{1+c}{(1+\chi)^2} \tag{60}$$

and defined $\lambda = \sqrt{\frac{\alpha}{1+c}}$. This expression for c agrees with the replica result (44). For the fraction ϕ of frozen agents one obtains

$$\phi = (1-f)\langle\theta(|z|-\gamma)\rangle + f = 1 - (1-f)\,\mathrm{erf}\frac{\lambda}{\sqrt{2}}. \tag{61}$$

In figure 3 analytical results for c and ϕ are compared with simulations.

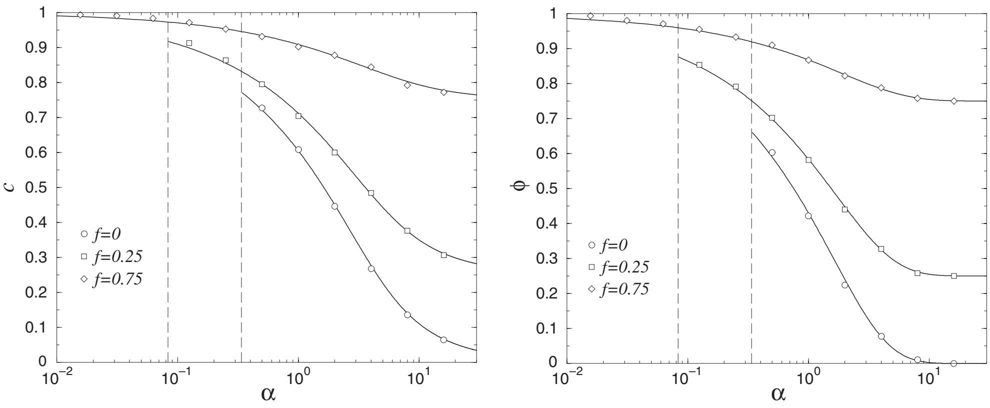

Figure 3. Persistent autocorrelation c (left) and fraction of frozen agents ϕ (right) for various f. Lines correspond to the analytic solutions from (59) and (61), markers are the results from numerical simulations. Vertical lines give, for $f < 1/2$, the positions of the critical points α_c below which the stationary state (hence c and ϕ) depends on initial conditions.

The 'susceptibility' (57) can instead be calculated from the formula

$$\chi = (1-f)\frac{\langle sz\rangle_{\min}}{\sqrt{\alpha}\langle z^2\rangle} - f\frac{\langle sz\rangle_{\text{maj}}}{\sqrt{\alpha}\langle z^2\rangle} \tag{62}$$

where $\langle sz\rangle_{\min}$ (resp. $\langle sz\rangle_{\text{maj}}$) denotes the average over z of the product sz for minority (resp. majority) agents. The above expression follows directly from the fact that, because the noise term and the external field enter (54) in the same way (apart from a $\sqrt{\alpha}$ factor), response functions for minority (resp. majority) agents can be obtained as $\alpha^{-1/2}\langle\partial\,\text{sign}[y(t)]/\partial z(t')\rangle_{-1}$ (resp. $\alpha^{-1/2}\langle\partial\,\text{sign}[y(t)]/\partial z(t')\rangle_{1}$), after an integration by parts and a time average [17]. For the minority part, $sz = |z|$ for $|z| > \gamma$ and $sz = z^2/\gamma$ otherwise, so that

$$\langle sz\rangle_{\min} = \langle\theta(|z|-\gamma)|z|\rangle + \left\langle\theta(\gamma-|z|)\frac{z^2}{\gamma}\right\rangle = \frac{1+c}{\sqrt{\alpha}(1+\chi)}\,\text{erf}\,\frac{\lambda}{\sqrt{2}}. \tag{63}$$

To calculate the majority part, one must fix the value of sz for $-\gamma \leqslant z \leqslant \gamma$, where s can be either +1 or -1 (for $|z| > \gamma$ one has $sz = |z|$ in any case). The choice is apparently arbitrary so that, in principle, there are several possibilities. While it is clear that each of them describes a different 'freezing' situation for the individual agents, for instance concerning the growth of their respective preferences, it is not completely clear to us which is the more appropriate in general. We concentrate here on two extreme cases.

First, we assume that $s = \text{sign}(z)$. This assumption is the most natural for continuity reasons. One has

$$\langle sz\rangle_{\text{maj}} = \langle|z|\rangle = \sqrt{\frac{2}{\pi}}\sqrt{\frac{1+c}{(1+\chi)^2}}. \tag{64}$$

This leads to

$$\frac{\alpha\chi}{1+\chi} = (1-f)\,\text{erf}\,\frac{\lambda}{\sqrt{2}} - f\lambda\sqrt{\frac{2}{\pi}}. \tag{65}$$

χ diverges (hence FIR is violated and ergodicity is broken) when the fraction $\overline{\phi} = 1-\phi$ of fickle agents satisfies $\overline{\phi} = \alpha + f\lambda\sqrt{2/\pi}$ or, equivalently, at the critical values of α given by the equation

$$\alpha_c(f) = (1-f)\,\text{erf}(x) - \frac{2fx}{\sqrt{\pi}} \tag{66}$$

where x is the solution of

$$2 - (1-f)\,\mathrm{erf}(x) - \frac{1-f}{x\sqrt{\pi}}\,\mathrm{e}^{-x^2} + \frac{f}{x\sqrt{\pi}} = 0. \tag{67}$$

Equations (65)–(67) are in full agreement with the replica results of section 3.

Another possibility is to calculate $\langle sz\rangle_{\mathrm{maj}}$ without making any special assumption on s for $-\gamma \leqslant z \leqslant \gamma$. This brings us to a situation where (64)–(66) are substituted respectively by

$$\langle sz\rangle_{\mathrm{maj}} = \langle\theta(z+\gamma)z\rangle - \langle\theta(\gamma-z)z\rangle = \mathrm{e}^{-\lambda^2/2}\sqrt{\frac{2}{\pi}}\sqrt{\frac{1+c}{(1+\chi)^2}} \tag{68}$$

$$\frac{\alpha\chi}{1+\chi} = (1-f)\,\mathrm{erf}\frac{\lambda}{\sqrt{2}} - f\lambda\sqrt{\frac{2}{\pi}}\,\mathrm{e}^{-\lambda^2/2} \tag{69}$$

$$\alpha_c(f) = (1-f)\,\mathrm{erf}(x) - \frac{2fx}{\sqrt{\pi}}\,\mathrm{e}^{-x^2} \tag{70}$$

where x now solves

$$2 - (1-f)\,\mathrm{erf}(x) - \frac{1-2f}{x\sqrt{\pi}}\,\mathrm{e}^{-x^2} = 0. \tag{71}$$

The value of $\overline{\phi}$ at which χ diverges is now $\overline{\phi} = \alpha + f\lambda\,\mathrm{e}^{-\lambda^2/2}\sqrt{2/\pi}$. Note that the extra exponential factor one obtains in this way does not change numerical results for α_c significantly (indeed, it changes by less than 0.0015). This is quite remarkable. However, one must note also that for a purely majority game (recalling that $\chi_2 = -\chi$) one gets for the susceptibility

$$\chi_2 = \frac{\mathrm{e}^{-\alpha/4}/\sqrt{\alpha\pi}}{1+\mathrm{e}^{-\alpha/4}/\sqrt{\alpha\pi}} \tag{72}$$

instead of the Hopfield-like formula (48). In both cases, $\chi_2 \to \infty$ when $\alpha \downarrow 0$, but the corresponding pictures (and volatilities, see below) are slightly different. In the following, we stick to the latter formula mainly because it provides a better agreement with numerical results for σ^2 at $f = 1$, but the question of which susceptibility describes a pure majority game more accurately is an important point that might deserve further investigation.

For the stationary volatility, which reads [17]

$$\sigma^2 = \frac{1}{2}\lim_{t\to\infty}[(\mathsf{I}+\mathsf{G})^{-1}(\mathsf{E}+\mathsf{C})(\mathsf{I}+\mathsf{G}^T)^{-1}]_{tt} \tag{73}$$

one can use the approximate method of [17] to derive an expression in terms of the persistent parameters χ and ϕ, which holds for $\alpha > \alpha_c$:

$$\sigma^2 = \frac{1+\phi}{2(1+\chi)^2} + \frac{1}{2}(1-\phi). \tag{74}$$

Solving for χ, ϕ and c for different f and substituting one obtains the volatility branches displayed in figure 1, which are again in excellent agreement with simulations.

5. Summary and outlook

To summarize, we have studied the mixed majority–minority game with random external information. Neglecting 'retrieval' (i.e. the possibility that trend-followers flock in the presence of a particular information pattern), we have first calculated the stationary state of the dynamics from a static approximation via a negative-replica theory. Then we have solved the dynamics using generating functional methods. The two approaches match nicely and agree with numerical results for the macroscopic observables σ^2 and H in a satisfactory way. Our results also indicate that when fundamentalists outnumber trend-followers, the macroscopic behaviour of the system ('phase transition' with ergodicity breaking from an inefficient phase at high α to an efficient one at low α) can be explained by the onset of anomalous response, that is by a divergence of the integrated response, as in the pure minority game. We have calculated the line of critical points in the (f, α) plane showing that the inefficient phase gets larger as f increases. When trend-followers dominate, instead, the system is always inefficient and low volatility states disappear. As a byproduct, we have provided an approximate static and dynamical solution of the majority game. A greater effort is nevertheless needed in order to incorporate the possibility of 'herding' in both the replica theory and the path-integral solution. We expect retrieval states to exist at low α for any $f > 0$. It is also likely that RSB occurs at very low α when $f > 0$ (in the pure minority game, RSB is known to take place for any non-zero market impact [21, 36, 37]).

Let us finally remark some aspects of the present model that can be criticized and hence improved. In the first place, all players can in principle win at the same time, which is a clearly unrealistic situation (albeit extremely unlikely in our disordered setup with $N \to \infty$). This means that we do not consider a situation where agents are competing for a scarce resource, or that, in other words, the presence of majority game players can turn a minority game from being a negative-sum game to a possibly positive-sum game. The nature of our game is hence totally different from that of the pure minority game, and one may even question the effectiveness of σ^2 as a measure of global efficiency. We have stuck to σ^2 because it measures the amplitude of the relevant aggregate quantity of our model (namely the excess demand A) and is therefore what most resembles the 'volatility' in market models with a price dynamics. This is the customary magnitude of market fluctuations, which is a widely accepted measure even for real markets, because after all it provides a quantitative and yet intuitive understanding of how the presence of trend-followers alters the macroscopic behaviour of the system (see [8] for an example of a detailed market model where this effect is analysed).

The second point concerns time scales. In a market a large buy rush today is justified by the belief that *tomorrow* the price will rise again so that for instance one will be able to sell at a higher price. So in a majority game it would perhaps be more correct to measure the effectiveness of a trading decision made today by what the payoff will be tomorrow [5, 6]. In other words, a player making a trading decision $a_i(n)$ at round n should receive a payoff $u_i(n+1) = a_i(t)A(n+1)$ at round $n+1$. Instead, in our model, his payoff is $u_i(n) = a_i(n)A(n)$.

A further point is related to our use of random external information. It is known that trend-following behaviour is self-enforcing, in the sense that if everyone believes that a certain stock price will rise and buys some stocks, the price will actually rise. This scenario might repeat for long times, suggesting that bubbles are strongly connected to the fact that agents react to the real market history, rather than to particular initial biases in their preferences. So for a majority game substituting the latter with random information might be a hazardous, though useful, assumption. On the other hand, it should also be noted that in our setting where agents are not allowed to switch from being trend-followers to being contrarians one can also partly justify the use of random history: in a real market model it is the price history which, at certain times, 'creates' a flock of trend-followers; in our model we do not need such a mechanism because we have it fixed in the model definition.

Finally, we stress again that our aim was not to to build a realistic model of a market, but a simple abstraction for a system where trend-followers and fundamentalists interact, which is analytically tractable and possesses interesting non-trivial features that go in the direction of the phenomena observed in real markets. More realistic and complex market models should take into account the dynamics of the preferences *and* the dynamics of the price, this last being completely absent in our model. In models of this sort (see for example [5–9]) the agents' behaviour and their payoff functions are based on the price history. As regards the agents' character, there may be different kinds of agents in such models such as speculators and producers, whose aims (i.e. utility functions) are assumed as different and who play on different time scales. Besides, it is reasonable that a single agent not only has different strategies of action, but also can behave sometimes as a fundamentalist and at other times as a trend-follower (or trend adverse) depending on the price history. It is in this case the global dynamics, i.e. the dynamics of the agents coupled with the dynamics of the price, that determines endogenously which strategies are played and consequently defines temporary groups of players (fundamentalists, trend-followers, etc). In this respect we understand what are at present the main limitations of our model: (a) we assume that no agent can switch from being fundamentalist to trend-follower, and (b) we postulate the same time horizons for all the agents. Future work may improve our analysis in this direction, both allowing for agents playing on different time scales and for looking at a generalization of the present model where the ϵ_i are dynamical (annealed) variables in order to give agents the possibility of changing their character.

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Appendix. Generating functional analysis

The disorder average of (51) is expected to generate two-time player-averaged functions of the s_i and $\widehat{y}_i$ variables only. We focus on

$$L_{tt'} = \frac{1}{N}\sum_{i=1,N} \widehat{y}_i(t)\widehat{y}_i(t') \tag{75}$$

$$Q_{tt'} = \frac{1}{N}\sum_{i=1,N} s_i(t)s_i(t') \tag{76}$$

$$K_{tt'} = -\frac{1}{N}\sum_{i=1,N} \epsilon_i s_i(t)\widehat{y}_i(t'). \tag{77}$$

The matrix K can be seen as formed by two components, for minority and majority agents, respectively:

$$\mathsf{K} = (1-f)\mathsf{K}_1 - f\mathsf{K}_2. \tag{78}$$

Forcing the above definitions inside $\overline{Z}$ via δ-functions with the proper N-scaling and assuming that $p(\boldsymbol{y}(0)) = \prod_{i=1,N} p(y_i(0))$, we find (with the shorthand $D(\mathsf{X},\widehat{\mathsf{X}}) = \prod_{tt'} \mathrm{d}X_{tt'}\,\mathrm{d}\widehat{X}_{tt'}/(2\pi)$)

$$\overline{Z[\psi]} = \int \mathrm{e}^{N(\Psi+\Omega+\Phi)} D(\mathsf{Q},\widehat{\mathsf{Q}}) D(\mathsf{L},\widehat{\mathsf{L}}) D(\mathsf{K},\widehat{\mathsf{K}}) \tag{79}$$

where $\Psi(\mathsf{Q},\widehat{\mathsf{Q}},\mathsf{L},\widehat{\mathsf{L}},\mathsf{K},\widehat{\mathsf{K}}) = \mathrm{i}\,\mathrm{Tr}[\widehat{\mathsf{Q}}^T\mathsf{Q}+\widehat{\mathsf{L}}^T\mathsf{L}+\widehat{\mathsf{K}}^T\mathsf{K}]$,

$$\Omega(\widehat{\mathsf{Q}},\widehat{\mathsf{L}},\widehat{\mathsf{K}}) = \frac{1}{N}\sum_{i=1,N}\log\int D(y,\widehat{y})p(y(0))\,\mathrm{e}^{\mathrm{i}\sum_t \widehat{y}(t)[y(t+1)-y(t)-\theta_i(t)]+y(t)\psi_i(t)} \times \mathrm{e}^{-\mathrm{i}\sum_{tt'}[s(t)\widehat{Q}_{tt'}s(t')+\widehat{y}(t)\widehat{L}_{tt'}\widehat{y}(t')-\epsilon_i s(t)\widehat{K}_{tt'}\widehat{y}(t')]} \tag{80}$$

and $\Phi(\mathsf{Q},\mathsf{L},\mathsf{K}) = F(\widehat{\boldsymbol{y}})$. To calculate the latter, it suffices to make use of the definitions of h_i and J_{ij} and to introduce, via δ-functions, the parameters

$$x_t^\mu = \sqrt{\frac{2}{N}}\sum_{i=1,N}s_i(t)\xi_i^\mu \qquad \text{and} \qquad w_t^\mu = -\sqrt{\frac{2}{N}}\sum_{i=1,N}\epsilon_i\widehat{y}_i(t)\xi_i^\mu. \tag{81}$$

It turns out that the relevant term for the disorder average is

$$\overline{\mathrm{e}^{\mathrm{i}\sqrt{2}\sum_{t\mu}w_t^\mu\Omega^\mu-\mathrm{i}\sqrt{\frac{2}{N}}\sum_{i\mu}\xi_i^\mu\sum_t[\widehat{x}_t^\mu s_i(t)-\widehat{y}_i(t)\epsilon_i\widehat{w}_t^\mu]}} = \mathrm{e}^{-\frac{1}{2}\sum_{tt'\mu}(w_t^\mu w_{t'}^\mu+\widehat{w}_t^\mu L_{tt'}\widehat{w}_{t'}^\mu+\widehat{x}_t^\mu Q_{tt'}\widehat{x}_{t'}^\mu+2\widehat{x}_t^\mu K_{tt'}\widehat{w}_{t'}^\mu)} \tag{82}$$

so that finally one has (with $D(z,\widehat{z}) = \prod_t \mathrm{d}x_t\,\mathrm{d}\widehat{x}_t/(2\pi)$)

$$\Phi(\mathsf{Q},\mathsf{L},\mathsf{K}) = \alpha\log\int D(x,\widehat{x})D(w,\widehat{w}) \times \mathrm{e}^{\mathrm{i}\sum_t(x_t\widehat{x}_t+w_t\widehat{w}_t+x_tw_t)-\frac{1}{2}\sum_{tt'}[w_tw_{t'}+\widehat{w}_tL_{tt'}\widehat{w}_{t'}^\mu+\widehat{x}_tC_{tt'}\widehat{x}_{t'}+2\widehat{x}_tK_{tt'}\widehat{w}_{t'}]} \tag{83}$$

where all integrals are from $-\infty$ to $+\infty$.

In the limit $N\to\infty$ the dominant contribution to $\overline{Z[\boldsymbol{\psi}]}$ comes from the saddle point described by the equations

$$\mathrm{i}\widehat{Q}_{tt'} = -\partial_{Q_{tt'}}\Phi \qquad \mathrm{i}\widehat{L}_{tt'} = -\partial_{L_{tt'}}\Phi \qquad \mathrm{i}\widehat{K}_{tt'} = -\partial_{K_{tt'}}\Phi \tag{84}$$

$$Q_{tt'} = \langle s(t)s(t')\rangle_* \qquad L_{tt'} = \langle\widehat{y}(t)\widehat{y}(t')\rangle_* \qquad K_{tt'} = -\langle\epsilon_i s(t)\widehat{y}(t')\rangle_* \tag{85}$$

where

$$\langle h(s,y,\widehat{y})\rangle_* = \frac{1}{N}\sum_{i=1,N}\frac{\int h(s,y,\widehat{y})M_i^{\epsilon_i}(s,y,\widehat{y})D(y,\widehat{y})}{\int M_i^{\epsilon_i}(s,y,\widehat{y})D(y,\widehat{y})} \tag{86}$$

with

$$M_i^{\epsilon_i}(s,y,\widehat{y}) = p(y(0))\,\mathrm{e}^{\mathrm{i}\sum_t\widehat{y}(t)[y(t+1)-y(t)-\theta_i(t)]+y(t)\psi_i(t)} \times \mathrm{e}^{-\mathrm{i}\sum_{tt'}[s(t)\widehat{Q}_{tt'}s(t')+\widehat{y}(t)\widehat{L}_{tt'}\widehat{y}(t')-\epsilon_i s(t)\widehat{K}_{tt'}\widehat{y}(t')]}. \tag{87}$$

It can be checked by a direct calculation (e.g. following [17]) that, at the relevant saddle point,

$$Q_{tt'} = C_{tt'} \equiv \frac{1}{N}\sum_{i=1,N}\overline{\langle s_i(t)s_i(t')\rangle_{\text{paths}}} \qquad \text{and} \qquad L_{tt'} = 0. \tag{88}$$

As for $K_{tt'}$, one can define $-\mathrm{i}\mathsf{K} = \mathsf{G}$ and see, for instance by taking the derivative of $\overline{\langle s_i(t)\rangle_{\text{paths}}}$ with respect to $\theta_i(t')$, that

$$\mathsf{G} = (1-f)\mathsf{G}_1 - f\mathsf{G}_2 \tag{89}$$

where G_1 is the response function of minority agents, with elements

$$G_{tt'}^{(1)} = \frac{1}{N_1}\sum_{i\in N_1}\frac{\partial}{\partial\theta_i(t')}\overline{\langle s_i(t)\rangle_{\text{paths}}} \tag{90}$$

and similarly G_2 is the response function of majority agents.

Setting the generating field ψ_i to zero and assuming that $\theta_i(t) = \theta(t)$, we can now treat minority agents ($\epsilon_i = -1$) and majority agents ($\epsilon_i = 1$) separately. We get

$$\Omega^\epsilon = \log \int \mathrm{e}^{\mathrm{i}\sum_t \widehat{y}(t)[y(t+1)-y(t)-\theta(t)]}\, \mathrm{e}^{-\mathrm{i}\sum_{tt'}[s(t)\widehat{C}_{tt'}s(t')+\widehat{y}(t)\widehat{L}_{tt'}\widehat{y}(t')-\epsilon s(t)\widehat{K}_{tt'}\widehat{y}(t')]}\, p(y(0))D(y, \widehat{y}) \tag{91}$$

where we set $\widehat{\mathsf{Q}} = \widehat{\mathsf{C}}$; the measure $M_i^{\epsilon_i}$ instead becomes

$$M^\epsilon(s, y, \widehat{y}) = p(y(0))\, \mathrm{e}^{-\mathrm{i}\sum_{tt'} s(t)\widehat{C}_{tt'}s(t')}\, \mathrm{e}^{-\mathrm{i}\sum_{tt'} \widehat{y}(t)\widehat{L}_{tt'}\widehat{y}(t')+\mathrm{i}\sum_t \widehat{y}(t)[y(t+1)-y(t)-\theta(t)+\epsilon\sum_{t'} \widehat{K}^T_{tt'}s(t)]}. \tag{92}$$

M^1 and M^{-1} represent majority and minority agents, respectively. The above $*$-average can now be conveniently recast as the sum averages over the minority and majority processes,

$$\langle\,\rangle_* = (1-f)\langle\,\rangle_{-1} + f\langle\,\rangle_1 \tag{93}$$

where the -1- and 1-averages are performed with the measures M^{-1} and M^1, respectively. The saddle-point equations for $\widehat{\mathsf{C}}$, $\widehat{\mathsf{L}}$ and $\widehat{\mathsf{K}}$ are identical to those found for the pure batch minority game [17]. The result is

$$\begin{aligned} &\widehat{C}_{tt'} = 0 \qquad \widehat{K}^T_{tt'} = -\alpha[(\mathsf{I}-\mathrm{i}\mathsf{K})^{-1}]_{tt'} \\ &\widehat{L}_{tt'} = -\tfrac{1}{2}\mathrm{i}\alpha[(\mathsf{I}-\mathrm{i}\mathsf{K})^{-1}(\mathsf{E}+\mathsf{C})(\mathsf{I}-\mathrm{i}\mathsf{K}^T)^{-1}]_{tt'} \end{aligned} \tag{94}$$

where $I_{tt'} = \delta_{tt'}$ and $E_{tt'} = 1$. Substituting these into M^ϵ one obtains

$$\begin{aligned} M^\epsilon(s, y, \widehat{y}) = p(y(0))\, &\mathrm{e}^{-\frac{1}{2}\alpha\sum_{tt'} \widehat{y}(t)[(\mathsf{I}-\mathrm{i}\mathsf{K})^{-1}(\mathsf{E}+\mathsf{C})(\mathsf{I}-\mathrm{i}\mathsf{K}^T)^{-1}]_{tt'}\widehat{y}(t')} \\ &\times \mathrm{e}^{\mathrm{i}\sum_t \widehat{y}(t)[y(t+1)-y(t)-\theta(t)-\alpha\epsilon\sum_{t'}[(\mathsf{I}-\mathrm{i}\mathsf{K})^{-1}]_{tt'}s(t)]}. \end{aligned} \tag{95}$$

Recalling that $\mathsf{K} = \mathrm{i}\mathsf{G}$, it turns out that the disorder-averaged correlation and response functions for minority and majority agents are obtained as averages over the coloured effective stochastic processes (54) with $\epsilon = -1$ and $\epsilon = 1$, respectively.

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